Flows and particles with shear-free and expansion-free velocities in $(\overline{L}_n, g)$- and Weyl's spaces

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Abstract

Conditions for the existence of flows with non-null shear-free and expansion-free velocities in spaces with affine connections and metrics are found. On their basis, generalized Weyl's spaces with shear-free and expansion-free conformal Killing vectors as velocities vectors of spinless test particles moving in a Weyl's space are considered. The necessary and sufficient conditions are found under which a free spinless test particle could move in spaces with affine connections and metrics on a curve described by means of an auto-parallel equation. In Weyl's spaces with Weyl's covector, constructed by the use of a dilaton field, the dilaton field appears as a scaling factor for the rest mass density of the test particle.

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1 Introduction

1.1 Weyl's spaces as special case of spaces with affine connections and metrics

In the last years Weyl's spaces have deserved some interest related to the possibility of using mathematical models of space-time different from (pseudo) Riemannian spaces without torsion (V_n-spaces) or with torsion (U_n-spaces)[1] $\div$ [4]. On the one side, Weyl's spaces appear as a generalization of V_n- and U_n-spaces. On the other side, they are special cases of spaces with affine connections and metrics. The use of spaces with affine connections and metrics as models of space-time has been critically evaluated from different points of view [5], [6]. But recently, it has been proved that in spaces with contravariant and covariant affine connections (whose components differ only by sign) and metrics [(L_n, g)-spaces] as well as in spaces with contravariant and covariant affine connections (whose components differ not only by sign) and metrics [$(\overline{L}_n, g)$-spaces] [7] [8]

- the principle of equivalence holds [9]$\div$[16],

- special types of transports (called Fermi-Walker transports) [17] $\div$ [19] exist which do not deform a Lorentz basis, and therefore, the low of causality is not abused in (L_n, g)- and $(\overline{L}_n, g)$-spaces if one uses a Fermi-Walker transport instead of a parallel transport (used in a V_n-space).

- there also exist other types of transports (called conformal transports) [20], [21] under which a light cone does not deform,

- the auto-parallel equation can play the same role in (L_n, g)- and $(\overline{L}_n, g)$-spaces as the geodesic equation does in the Einstein theory of gravitation (ETG) in V_n-spaces [22], [23], where the geodesic equation is identical with the auto-parallel equation.

On this basis, many of the differential-geometric constructions used in the ETG in V_4-spaces could be generalized for the cases of (L_n, g)- and $(\overline{L}_n, g)$-spaces, and especially for Weyl's spaces without torsion (W_n or $\overline{W}_n$-spaces) or in Weyl's spaces with torsion (Y_n- or $\overline{Y}_n$-spaces) as special cases of (L_n, g)- or $(\overline{L}_n, g)$-spaces. Bearing in mind this background, a question arises about possible physical applications and interpretation of mathematical constructions from ETG generalized for Weyl's spaces with affine connections and metrics. On the other side, it is well known that every classical field theory over spaces with affine connections and metrics could be considered as a theory of continuous media in these spaces [24] $\div$ [27]. On this ground, notions of the continuous media mechanics (such as deformation velocity and acceleration, shear velocity and acceleration, rotation velocity and acceleration, expansion velocity and acceleration) [28] have been used as invariant characteristics for spaces admitting vector fields with special kinematic characteristics [29], [30].

The existence of a flow with shear-free and expansion-free velocity is of great importance for continuous media mechanics in the relativistic and non-relativistic case as well as for hydrodynamics as a part of it [28]. Such type of a flow exists when the flow does not change its form and volume, i.e. it does not deform, expand or shrink (contract). Flows with these properties are considered as stable during their motion in the space or in the space-time.

The main task of this paper is the investigation of Weyl's spaces with respect to their ability to admit flows and spinless test particles with velocities that are shear-free and expansion-free vector fields. On this basis, conditions for the existence of shear-free and expansion-free non-null (non-isotropic) vector fields in spaces with affine connections and metrics are found and then specialized for Weyl's spaces. At the same time, a possible interpretation of a dilaton field, appearing in the structure of special types of Weyl's spaces, is found on the basis of the auto-parallel equation describing the motion of a free spinless test particle in these types of spaces.

In Section 2, the notions of relative velocity, shear velocity, and expansion velocity are considered. The equivalence of the action of the Lie differential operator and of the covariant differential operator on the invariant volume element generates restrictions to a vector field along which these operators act and respectively to the existing in the manifold metrics. It is shown that the equivalence condition appears as a condition for the existence of shear-free and expansion-free non-null vector fields as velocities vector fields of flows or particles in spaces with affine connections and metrics. The same condition in a Weyl's space appears as a condition for the existence of a shear-free and expansion-free conformal Killing vector field of special type. Sufficient conditions are found under which in special type of Weyl's spaces non-null auto-parallel, shear-free, and expansion-free conformal Killing vector fields could exist as velocities vector fields of a flow or of spinless test particles. In Section 3 the auto-parallel equation in Weyl's spaces is discussed as an equation for describing the motion of a free moving spinless test particle. Concluding remarks

comprise the final Section 4. The most considerations are given in details (even in full details) for those readers who are not familiar with the investigated problems.

1.2 Abbreviations, definitions, and symbols

In the further considerations in this paper we will use the following abbreviations, definitions and symbols:

$:=$ means by definition.

Point " $\cdot$" is used as a symbol for standard multiplication in the field of real (or complex) numbers, e.g. $a \cdot b \in \mathbf{R}$.

The point " ." is used as a symbol for symmetric tensor product, e.g. $u.v = \frac{1}{2} \cdot (u \otimes v + v \otimes u)$.

"$\wedge$" is the symbol for a wedge product, e.g. $u \wedge v = \frac{1}{2} \cdot (u \otimes v - v \otimes u)$.

M is a symbol for a differentiable manifold with $\dim M = n$. $T(M) := \cup_{x \in M} T_x(M)$ and $T^*(M) := \cup_{x \in M} T^*_x(M)$ are the tangent and the cotangent spaces at M respectively.

$(\overline{L}_n, g)$, $\overline{Y}_n$, $\overline{U}_n$, and $\overline{V}_n$ are spaces with contravariant and covariant affine connections and metrics whose components *differ not only by sign* [7]. In such type of spaces the non-canonical contraction operator S acts on a contravariant basic vector field e_j (or ∂_j) $\in \{e_j$ (or $\partial_j)\} \subset T(M)$ and on a covariant basic vector field e^i (or dx^i) $\in \{e^i$ (or $dx^i)\} \subset T^*(M)$ in the form

$$\begin{array}{rcllll} S & : & (e^i, e_j) \longrightarrow S(e^i, e_j) := S(e_j, e^i) := f^i{}_j \ , & & & \\ f^i{}_j & \in & C^r(M) \ , & r \geqq 2 \ , & \det(f^i{}_j) \neq 0, & \\ \exists \ \ f_i{}^k & \in & C^r(M), & r \geqq 2 \ : & f^i{}_j \cdot f_i{}^k := g^k_j . & \end{array}$$

The functions $f^i{}_j = f^i{}_j(x^k)$ fulfil equations

$$f^i{}_{j,k} = \Gamma^l_{jk} \cdot f^i{}_l + P^i_{lk} \cdot f^l{}_j \quad , \qquad f^i{}_{j,k} = \partial_k f^i{}_j = \frac{\partial f^i{}_j}{\partial x^k} \ ,$$

related to the both affine connections Γ and P with components Γ^i_{jk} and P^i_{jk} in the co-ordinate bases $\{\partial_i\}$ and $\{dx^i\}$ respectively.

In these spaces, for example, $g(u) = g_{ik} \cdot f^k{}_j \cdot u^j \cdot dx^i := g_{i\overline{j}} \cdot u^j \cdot dx^i = g_{ij} \cdot u^{\overline{j}} \cdot dx^i := u_i \cdot dx^i$, $g(u,u) = g_{kl} \cdot f^k{}_i \cdot f^l{}_j \cdot u^i \cdot u^j := g_{\overline{ij}} \cdot u^i \cdot u^j = g_{ij} \cdot u^{\overline{i}} \cdot u^{\overline{j}} = u_j \cdot u^{\overline{j}} := u_{\overline{i}} \cdot u^i$, $g^{\overline{ij}} \cdot g_{jk} = \delta^i_k = g^i_k$, $g_{\overline{ik}} . g^{kj} = g^j_i$. The components $\delta^i_j := g^i_j$ ($|= 0$ for $i \neq j$ and $|= 1$ for $i = j$) are the components of the Kronecker tensor $Kr := g^i_j \cdot \partial_i \otimes dx^j$.

(L_n, g), Y_n, U_n, and V_n are spaces with contravariant and covariant affine connections and metrics whose components *differ only by sign* [31], [32]. In such type of spaces the canonical contraction operator $S := C$ acts on a contravariant basic vector field e_j (as a non-co-ordinate, non-holonomic contravariant basic vector field) [or ∂_j (as a co-ordinate, holonomic contravariant basic vector field)] $\in \{e_j$ (or $\partial_j)\} \subset T(M)$ and on a covariant basic vector field e^i (as a non-co-ordinate, non-holonomic covariant basic vector field) [or dx^i as a co-ordinate, holonomic covariant basic vector field)] $\in \{e^i$ (or $dx^i)\} \subset T^*(M)$ in the form

$$C : (e^i, e_j) \longrightarrow C(e^i, e_j) := C(e_j, e^i) := \delta^i_j := g^i_j \ .$$

In these spaces, for example, $g(u) = g_{ik} \cdot g^k_j \cdot u^j \cdot dx^i := g_{ij} \cdot u^j \cdot dx^i = u_i \cdot dx^i$, $g(u,u) = g_{kl} \cdot g^k_i \cdot g^l_j \cdot u^i \cdot u^j := g_{ij} \cdot u^i \cdot u^j = u_i \cdot u^i$.

The different types of spaces with affine connections and metrics with respect to their properties related to the contraction operator S (or C) and to the metric g could be given roughly in the following scheme

Space	Contraction operator	Components of the contravariant and covariant affine connections Γ and P	Covariant derivative of the metric with respect to the affine connections
$(\overline{L}_n, g)$-space	$S = f^i{}_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} \neq -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$ $P^i_{jk} \neq P^i_{kj}$	$g_{ij;k} \neq 0$
(L_n, g)-space	$S = C = g^i_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} = -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$	$g_{ij;k} \neq 0$
$\overline{Y}_n$-space (Weyl′s space with torsion)	$S = f^i{}_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} \neq -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$ $P^i_{jk} \neq P^i_{kj}$	$g_{ij;k} = \frac{1}{n} \cdot Q_k \cdot g_{ij}$
Y_n-space (Weyl′s space with torsion)	$S = C = g^i_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} = -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$	$g_{ij;k} = \frac{1}{n} \cdot Q_k \cdot g_{ij}$
$\overline{U}_n$-space (Pseudo-Riemannian space with torsion)	$S = f^i{}_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} \neq -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$ $P^i_{jk} \neq P^i_{kj}$	$g_{ij;k} = 0$
U_n-space (Pseudo-Riemannian space with torsion)	$S = C = g^i_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} = -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$	$g_{ij;k} = 0$
$\overline{V}_n$-space (Pseudo-Riemannian space without torsion)	$S = f^i{}_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} \neq -P^i_{jk}$ $\Gamma^i_{jk} \neq \Gamma^i_{kj}$ $P^i_{jk} = P^i_{kj}$	$g_{ij;k} = 0$
V_n-space (Pseudo-Riemannian space without torsion)	$S = C = g^i_j \cdot \partial_i \otimes dx^j$	$\Gamma^i_{jk} = -P^i_{jk}$ $\Gamma^i_{jk} = \Gamma^i_{kj}$	$g_{ij;k} = 0$

Remark. All results found for $(\overline{L}_n, g)$-spaces could be specialized for (L_n, g)-spaces by omitting the bars above or under the indices.

∇_u is the covariant differential operator acting on the elements of the tensor algebra $\mathcal{T}$ over M. The action of ∇_u is called covariant differentiation (covariant transport) along a contravariant vector field u, for instance,

$$\nabla_u v := v^i{}_{;j} \cdot u^j \cdot \partial_i = (v^i{}_{,j} + \Gamma^i_{kj} \cdot v^k) \cdot u^j \cdot \partial_i \ , \qquad v \in T(M) \ , \tag{1}$$

where $v^i{}_{,j} := \partial v^i / \partial x^j$ and Γ^i_{jk} are the components of the contravariant affine connection Γ in a contravariant co-ordinate basis $\{\partial_i\}$. The result $\nabla_u v$ of the action of ∇_u on a tensor field $v \in \otimes^k_l(M)$ is called covariant derivative of v along u. For covariant vectors and tensor fields an analogous relation holds, for instance,

$$\nabla_u w = w_{i;j} \cdot u^j \cdot dx^i = (w_{i,j} + P^l_{ij} . w_l) \cdot u^j \cdot .dx^i \quad , \quad w \in T^*(M) \ . \tag{2}$$

where P^l_{ij} are the components of the covariant affine connection P in a covariant co-ordinate basis $\{dx^i\}$. For (L_n, g), Y_n, W_n, U_n, and V_n-spaces $P^l_{ij} = -\Gamma^l_{ij}$.

$\pounds_u$ is the Lie differential operator [7] acting on the elements of the tensor algebra $\mathcal{T}$ over M. The action of $\pounds_u$ is called dragging-along a contravariant vector field u. The result $\pounds_u v$ of the action of $\pounds_u$ on a tensor field v is called Lie derivative of v along u. The Lie derivative of a contravariant vector ξ along a contravariant vector u could be written in the form

$$\pounds_u \xi = \nabla_u \xi - \nabla_\xi u - T(u, \xi) \ .$$

The contravariant vector $T(u, \xi)$ is called contravariant torsion vector. The tensor $T = T_{ij}{}^k \cdot dx^i \wedge dx^j \otimes \partial_k$ is called contravariant torsion tensor. Its components $T_{ij}{}^k$ in a co-ordinate basis have the form

$$T_{ij}{}^k := \Gamma^k_{ji} - \Gamma^k_{ij} \ .$$

The n-form $d\omega := \frac{1}{n!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_n} \cdot dx^{i_1} \wedge \dots \wedge x^{i_n}$, where $d_g := \det(g_{ij}) < 0$, $\varepsilon_{i_1 \dots i_n}$ are the components of the full antisymmetric Levi-Civita symbol, is called invariant volume element in M.

Line of a flow is a line with tangent vector at each of its points collinear (or identical) with the velocity of a material point with the corresponding position [28].

Trajectory of a particle. A line of a flow could be considered separately as the trajectory of a particle moving in a $(\overline{L}_n, g)$-space.

Flow is a congruence (family) of flow's lines.

For the further considerations we need to recall some properties of the Weyl's spaces and the conformal Killing vector fields.

1.3 Properties of a Weyl's space with affine connections and metrics

Usually, a Weyl's space without torsion and with contraction operator $S = C$ is defined by means of the condition for the vanishing of the covariant derivative of a metric $\overline{g} := \varphi \cdot g$ conformal to a given metric g, where $\varphi \in C^r(M)$,

$$\nabla_u \overline{g} = \nabla_u(\varphi \cdot g) := 0 \quad \text{for} \quad \forall u \in T(M) \ .$$

The introduced definition is equivalent to the definition for a Weyl's space as a space fulfilling the condition

$$\nabla_u g = q_u \cdot g \quad , \qquad q_u = -u(log \varphi) \quad , \qquad \varphi \in C^r(M) \quad ,$$

and leads automatically to the proposition 1 and 2 considered below.

A more general definition for a Weyl's space with affine connections and metrics (which includes $\overline{Y}_n$- and Y_n-spaces) could be introduced on the basis of a recurrent relation for the metric g.

1. Let us now consider the condition for the existence of a Weyl's space with affine connections and metrics on the basis of the following definition

Definition 1. A Weyl's space is a differentiable manifold M with $\dim M := n$, provided with affine connections Γ and P (with $P \neq -\Gamma$ or $P = -\Gamma$) and a metric g with covariant derivative of g along an arbitrary given contravariant vector field $u \in T(M)$ in the form

$$\nabla_u g := \frac{1}{n} \cdot Q_u \cdot g \ . \tag{3}$$

The existence condition is a recurrent relation for the metric g. Here

$$\begin{aligned} Q_u &= Q_j \cdot u^j \ , \quad Q := Q_j \cdot dx^j \ , \\ Q_j &: = \underline{Q}_k \cdot f^k{}_j := \underline{Q}_{\overline{j}} \text{ for } P \neq -\Gamma \\ Q_j &\equiv Q_j \text{ for } P = -\Gamma \ . \end{aligned} \tag{4}$$

The covariant vector field (1-form) $\overline{Q} := \frac{1}{n} \cdot Q_j \cdot dx^j = \frac{1}{n} \cdot Q$ is called Weyl's covariant (covector) field. If Q is an exact form, i. e. if $Q = -d\overline{\varphi} = -\overline{\varphi}_{,j} \cdot dx^j$ with $Q_j = -\overline{\varphi}_{,j}$, $\overline{\varphi} \in C^r(M)$, $r \geqq 2$, then for a contravariant vector field $u := d/d\tau$ the invariant Q_u could be written in the form $Q_u = -d\overline{\varphi}/d\tau$. The scalar field $\overline{\varphi}$ is called dilaton field. The reason for this notation follows from the properties of the Weyl's spaces considered below.

After contracting $\nabla_u g$ and g from the last equation with $\overline{g} = g^{ij} \cdot \partial_i . \partial_j$ by both basic vector fields, i.e. after finding the relations

$$\overline{g}[\nabla_u g] = g^{\overline{ij}} \cdot g_{ij;k} \cdot u^k \qquad \text{and} \qquad \overline{g}[g] = g^{\overline{kl}} . g_{kl} = n = \dim M \quad , \tag{5}$$

it follows for Q_u

$$Q_u = \overline{g}[\nabla_u g] = 2 \cdot \ \overline{f}_u \quad , \qquad \overline{f}_u = \frac{1}{2} \cdot Q_u \qquad . \tag{6}$$

Therefore, for Weyl's spaces, we have the recurrent relation for the invariant volume element $d\omega$

$$\nabla_u(d\omega) = \frac{1}{2} \cdot Q_u . d\omega \qquad . \tag{7}$$

2. Parallel transports over Weyl's spaces are at the same time conformal transports. This means that if $\nabla_u \xi = 0$ and $\nabla_u \eta = 0$, then $ul_\xi = (1/2n) \cdot Q_u \cdot l_\xi$, $ul_\eta = (1/2n) \cdot Q_u \cdot l_\eta$, and $u[\cos(\xi, \eta)] = 0$, where $l_\xi :=\mid g(\xi, \xi) \mid^{1/2}$, $l_\eta := \mid g(\eta, \eta) \mid^{1/2}$, $\cos(\xi, \eta) := [g(\xi, \eta)]/(l_\xi \cdot l_\eta)$. If $u = d/d\tau$, then [33]

$$\frac{dl_\xi}{d\tau} = \frac{1}{2n} \cdot Q_u \cdot l_\xi \ , \qquad \frac{dl_\eta}{d\tau} = \frac{1}{2n} \cdot Q_u \cdot l_\eta \quad , \tag{8}$$

and therefore,

$$\begin{aligned} l_\xi &= l_{\xi 0} \cdot \exp[\frac{1}{2n} \cdot \int Q_j \cdot dx^j] \ , \ \ l_\eta = l_{\eta 0} \cdot \exp[\frac{1}{2n} \cdot \int Q_j \cdot dx^j] \ , \\ l_{\xi 0} &= \text{const.}, \, l_{\eta 0} = \text{const.}, \qquad \cos(\xi, \eta) = \text{const. along } \tau(x^k). \end{aligned} \tag{9}$$

Furthermore, if $Q_j = -n \cdot \overline{\varphi}_{,j}$ and respectively $Q_u = -n \cdot d\overline{\varphi}/d\tau$, then the equation for l_ξ obtains in this case the simple form

$$\frac{dl_\xi}{d\tau} = -\frac{1}{2} \cdot \frac{d\overline{\varphi}}{d\tau} \cdot l_\xi \ . \tag{10}$$

The solution for l_ξ could easily be found as

$$l_\xi = l_{\xi 0} \cdot e^{-\frac{1}{2} \cdot \overline{\varphi}} \qquad . \tag{11}$$

The scalar field $\overline{\varphi}$ [as an invariant function $\overline{\varphi} \in \otimes^0{}_0(M)$] appears as a gauge factor changing the length of the vector ξ. This is the reason for calling the scalar field $\overline{\varphi}$ *dilaton field* in a Weyl's space.

3. The metric in a Weyl's space has properties which can be formulated in the following two propositions:

Proposition 1. [34] A metric $\widetilde{g}$ conformal to a Weyl's metric g is also a Weyl's metric. In other words, if $\widetilde{g} = e^{\varphi} \cdot g$ with $\nabla_\xi g = \frac{1}{n} \cdot Q_\xi \cdot g$ for $\forall \xi \in T(M)$, then $\nabla_\xi \widetilde{g} = \frac{1}{n} \cdot \widetilde{Q}_\xi \cdot \widetilde{g}$.

The proof is trivial.

Therefore, all Weyl's metrics belong to the set of all metrics conformal to a Weyl's metric.

Let the square ds^2 of the line element ds for a Weyl's metric g in $\overline{W}_n$- (or $\overline{Y}_n$)-spaces and in W_n- (or Y_n)-spaces be given in the forms respectively

$$ds^2 = g_{\overline{ij}} \cdot dx^i \cdot dx^j \quad , \qquad ds^2 = g_{ij} \cdot dx^i \cdot dx^j \quad .$$

Then, the square $d\widetilde{s}^2$ of the line element $d\widetilde{s}$ for a conformal to the Weyl's metric g will have the forms respectively

$$d\widetilde{s}^2 = \widetilde{g}_{\overline{ij}} \cdot dx^i \cdot dx^j = e^{\varphi}.ds^2 \quad , \qquad d\widetilde{s}^2 = \widetilde{g}_{ij} \cdot dx^i \cdot dx^j = e^{\varphi}.ds^2 \quad .$$

The invariant function $\varphi = \varphi(x^k) \in \otimes^0_0(M) \subset C^r(M)$, $r \geqq 2$, is also called *dilaton field* because of its appearing as a gauge factor changing the line element in a Weyl's space.

For $\pm\widetilde{l}^2_d = \widetilde{g}(d,d) = d\widetilde{s}^2 = \widetilde{g}_{\overline{ij}} \cdot dx^i \cdot dx^j$ with $d^i := dx^i$ we have

$$\pm\widetilde{l}^2_d = d\widetilde{s}^2 = e^{\varphi}.ds^2 = \pm e^{\varphi}.l^2_d \ .$$

On the other side, by the use of the expression (9) for the length $\widetilde{l}_d$ we find that

$$\begin{aligned} \widetilde{l}^2_d &= \widetilde{l}^2_{d0}.\exp[\frac{1}{n} \cdot \int \widetilde{Q}_j \cdot dx^j] = \widetilde{l}^2_{d0} \cdot \exp[\frac{1}{n} \cdot \int (n \cdot \varphi_{,j} + Q_j) \cdot dx^j] = \\ &= \widetilde{l}^2_{d0} \cdot \exp\varphi \cdot \exp[\frac{1}{n} \cdot \int Q_j \cdot dx^j] = e^{\varphi} \cdot l^2_d \quad , \qquad \widetilde{l}^2_{d0} := l^2_{d0} = \text{ const.} \end{aligned}$$

For $Q_j = -n \cdot \overline{\varphi}_{,j}$, it follows that

$$\widetilde{l}^2_d = \widetilde{l}_{d0} \cdot e^{\varphi - \overline{\varphi}} \quad .$$

If $\widetilde{l}^2_d$ does not change along the vector field u, then $\varphi = \overline{\varphi}$ and we have only one dilation field $\varphi = \varphi(x^k) = \overline{\varphi}(x^k)$ which determines the conformal factor of the metric $\widetilde{g}$, conformal to a Weyl's metric g, as well as the Weyl's covector Q. In this case the metric $\widetilde{g}$ appears as a Riemannian metric because of $\widetilde{Q}_j = 0$. If $\varphi \neq \overline{\varphi}$, then the metric $\widetilde{g}$, conformal to g is again a Weyl's metric with $\widetilde{Q}_j = -n \cdot \widetilde{\varphi}_{,j}$ and with $\widetilde{\varphi} = -(\overline{\varphi} - \varphi)$.

Proposition 2. The necessary and sufficient condition for a metric $\widetilde{g}$ conformal $(\widetilde{g} = e^{\varphi} \cdot g)$ to a Weyl's metric g [obeying the condition $\nabla_\xi g = \frac{1}{n} Q_\xi \cdot g$ for $\forall \xi \in T(M)$] to be a Riemannian metric $\widetilde{g}$ [obeying the condition $\nabla_\xi \widetilde{g} = 0$ for $\forall \xi \in T(M)$] is the condition

$$Q_\xi = -n \cdot (\xi\varphi) \quad , \qquad \xi \in T(M) \ , \qquad \varphi \in C^r(M) \ , \ r \geqq 2 \ . \tag{12}$$

The proof is trivial.

Corollary. All Riemannian metrics are conformal to a Weyl's metric in a Weyl's space with $Q_\xi = -n \cdot \xi\varphi$ for $\forall \xi \in T(M)$, $\varphi \in C^r(M)$, $r \geqq 2$, [or in a co-ordinate basis with $Q_k = -n \cdot \varphi_{,k}$].

1.4 Conformal Killing vectors

A conformal Killing vector field is defined by a recurrent equation.

Definition 2. A conformal Killing vector field is a contravariant vector field u obeying the recurrent equation

$$\pounds_u g = \lambda_u \cdot g \quad , \qquad \lambda_u \in \otimes^0_0(M) \subset C^r(M) \ . \tag{13}$$

The equation is called conformal Killing equation. After contracting $\pounds_u g$ and g from the last equation with $\overline{g} = g^{ij} \cdot \partial_i.\partial_j$ $[\partial_i.\partial_j := \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i)]$ by both basic vector fields, i.e. after finding the relations

$$\overline{g}[\pounds_u g] = g^{\overline{kl}} \cdot \pounds_u g_{ij} \qquad \text{and} \qquad \overline{g}[g] = g^{\overline{kl}}.g_{kl} = n = \dim M \quad , \tag{14}$$

it follows for λ_u

$$\lambda_u = \frac{1}{n} \cdot \overline{g}[\pounds_u g] = \frac{2}{n} \cdot \widetilde{f}_u \ . \tag{15}$$

On the other side, the results of the action of the covariant differential operator ∇_u and of the Lie differential operator $\pounds_u$ on the invariant volume element $d\omega$ are recurrent relations for $d\omega$ [7]

$$\nabla_u(d\omega) = \overline{f}_u \cdot d\omega \quad , \qquad \overline{f}_u := \frac{1}{2} \cdot \overline{g}[\nabla_u g] \in \otimes^0_0(M) \subset C^r(M) \ , \tag{16}$$

$$\pounds_u(d\omega) = \widetilde{f}_u \cdot d\omega \quad , \qquad \widetilde{f}_u := \frac{1}{2} \cdot \overline{g}[\pounds_u g] \in \otimes^0_0(M) \subset C^r(M) \ . \tag{17}$$

Therefore, the factor λ_u in a conformal Killing vector equation is related to the factor $\widetilde{f}_u$ by which the invariant volume element changes when dragged along the same vector field.

1.5 Problems and results

On the basis of the proposition 2 we can state that for *every* given Riemannian metric $\widetilde{g}$ from a Riemannian space and a *given scalar (dilaton) field* $\varphi(x^k)$ in this space we could generate a Weyl's metric g in a Weyl's space with the same affine connection as the affine connection in the Riemannian space. Vice versa, for every given Weyl's space with Weyl's covariant vector field Q constructed by a scalar (dilaton) field φ and a Weyl's metric g we could generate a Riemannian metric $\widetilde{g}$ in a Riemannian space with the same affine connection as in the corresponding Weyl's space.

Therefore, *every (metric) tensor-scalar theory of gravitation in a (pseudo) Riemannian space (with or without torsion) could be reformulated in a corresponding Weyl's space with Weyl's metric and dilaton field, generating the Weyl's covector in the Weyl's space* and vice versa: *every (metric) tensor-scalar theory in a Weyl's space with scalar (dilaton) field, generating the Weyl's covector, could be reformulated in terms of a (metric) tensor-scalar theory in the corresponding Riemannian space with the same affine connections as the affine connection in the Weyl's space.*

The last statement generates the idea of considering the motion of a flow or the motion of a spinless test particle in a Weyl's space under the following conditions:

(a) the acceleration of the particle (material element) of the flow is orthogonal to its velocity,

(b) the velocity of the particle (material element) of the flow is a non-null shear-free and expansion-free vector

(c) the equation of motion of the particle appears in a special case as an auto-parallel equation.

The kinematic characteristics of a vector field obeying the condition $\nabla_u g - \pounds_u g = 0$ can now be considered from a more general point of view, namely, for spaces with affine connections (which components differ not only by sign) and then specialized for Weyl's spaces.

2 Shear-free and expansion-free flows

Let us now consider conditions for the existence of a shear-free and expansion-free flow in a $(\overline{L}_n, g)$-space and especially in a Weyl's space. For this reason we need some facts about

- the notion of relative velocity, shear velocity, and expansion velocity,
- conformal Killing vectors,

- the properties of Weyl's spaces.

2.1 Deformation velocity, shear velocity, rotation velocity and expansion velocity

The notion of relative velocity in a V_n-space ($n = 4$) has been introduced by definition by Ehlers [29] as the velocity of a particle with respect to an other particle of its neighborhood in a flow. The physical interpretation of the notion of relative velocity in $(\overline{L}_n, g)$-spaces has been recently discussed in details in [28]. It has been shown that the relative velocity ${}_{rel}v$ is the velocity between two particles (matter elements) lying at a cross-section (hypersurface) orthogonal to the velocity u and having equal proper times.

If we introduce the relations

$$h_u := g - \frac{1}{e} \cdot g(u) \otimes g(u) \ , \qquad h_u = h_{ij} \cdot e^i.e^j \ , \qquad \overline{g} := g^{ij} \cdot e_i.e_j ,$$

$$\nabla_u \xi = \xi^i{}_{;j} \cdot u^j \cdot e_i \ , \qquad \xi^i{}_{;j} := e_j \xi^i + \Gamma^i_{kj} \cdot \xi^k \ , \qquad \Gamma^i_{kj} \neq \Gamma^i_{jk} \ ,$$

$$e := g(u,u) = g_{\overline{i}\overline{j}} \cdot u^i \cdot u^j = u_{\overline{i}} \cdot u^i \neq 0 \ , \qquad g(u) = g_{\overline{i}\overline{k}} \cdot u^k \cdot e^i = u_{\overline{i}} \cdot e^i \ ,$$

$$h_{ij} = g_{ij} - \frac{1}{e} \cdot u_i \cdot u_j \ , \qquad e_i.e_j := \frac{1}{2} \cdot (e_i \otimes e_j + e_j \otimes e_i) \ ,$$

$$h_u(\nabla_u \xi) = h_{i\overline{j}} \cdot \xi^j{}_{;k} \cdot u^k \cdot e^i \ ,$$

the relative velocity ${}_{rel}v$ [28], [35] could be represented in the form

$$\begin{aligned} {}_{rel}v &= \overline{g}[h_u(\nabla_u \xi)] = g^{ij} \cdot h_{\overline{j}\overline{k}} \cdot \xi^k{}_{;l} \cdot u^l \cdot e_i \quad , \\ e_i &= \partial_i \text{ (in a co-ordinate basis)}, \end{aligned} \tag{18}$$

where $\overline{g}[h_u(\xi)] := \xi_{\perp} = g^{ik} \cdot h_{\overline{k}\overline{l}} \cdot \xi^l \cdot e_i$ is called deviation vector field and (the indices in a co-ordinate and in a non-co-ordinate basis are written in both cases as Latin indices instead of Latin and Greek indices)

The relative velocity ${}_{rel}v$ could be written in a $(\overline{L}_n, g)$-space under the conditions $g(u, \xi) := l = 0$, $\pounds_\xi u = 0$, in the form [35], [36]

$${}_{rel}v = \overline{g}[d(\xi)] \ .$$

The covariant tensor field d is a generalization for $(\overline{L}_n, g)$-spaces of the well known *deformation velocity* tensor for V_n-spaces [30], [37]. It is usually represented by means of its three parts: the trace-free symmetric part, called *shear velocity* tensor (shear), the anti-symmetric part, called *rotation velocity* tensor (rotation) and the trace part, in which the trace is called *expansion velocity* (expansion) invariant. The physical interpretation of all parts of the deformation velocity tensor for the continuous media mechanics in $(\overline{L}_n, g)$-spaces is discussed in [28].

After some more complicated as for V_n-spaces calculations, the deformation velocity tensor d can be given in the form [35]

$$d = \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u \ . \tag{19}$$

The tensor σ is the *shear velocity* tensor (shear) ,

$$\begin{aligned} \sigma = {}_sE - {}_sP &= E - P - \frac{1}{n-1} \cdot \overline{g}[E - P] \cdot h_u = \sigma_{ij} \cdot e^i.e^j = \\ &= E - P - \frac{1}{n-1} \cdot (\theta_o - \theta_1) \cdot h_u \ , \end{aligned} \tag{20}$$

where

$$
{}_sE = E - \tfrac{1}{n-1} \cdot \overline{g}[E] \cdot h_u \ , \qquad \overline{g}[E] = g^{\overline{ij}} \cdot E_{ij} = \theta_o \ ,
$$

$$
E = h_u(\varepsilon)h_u \ , \qquad k_s = \varepsilon - m \ , \qquad \varepsilon = \tfrac{1}{2}(u^i{}_{;l} \cdot g^{lj} + u^j{}_{;l} \cdot g^{li}) \cdot e_i.e_j \ , \tag{21}
$$

$$
m = \tfrac{1}{2}(T_{lk}{}^i \cdot u^k \cdot g^{lj} + T_{lk}{}^j \cdot u^k \cdot g^{li}) \cdot e_i.e_j \ .
$$

The tensor ${}_sE$ is the *torsion-free shear velocity* tensor, the tensor ${}_sP$ is the *shear velocity* tensor *induced by the torsion*,

$$
{}_sP = P - \tfrac{1}{n-1} \cdot \overline{g}[P] \cdot h_u \ , \qquad \overline{g}[P] = g^{\overline{kl}} \cdot P_{kl} = \theta_1, \quad P = h_u(m)h_u \ ,
$$

$$
\theta_1 = T_{kl}{}^k \cdot u^l \ , \quad \theta_o = u^n{}_{;n} - \tfrac{1}{2e}(e_{,k} \cdot u^k - g_{kl;m} \cdot u^m \cdot u^{\overline{k}} \cdot u^{\overline{l}}) \ , \ \theta = \theta_o - \theta_1 \ . \tag{22}
$$

The invariant θ is the *expansion velocity,* the invariant θ_o is the *torsion-free expansion velocity,* the invariant θ_1 is the *expansion velocity induced by the torsion.*

The tensor ω is the *rotation velocity* tensor (rotation velocity),

$$
\begin{aligned}
\omega &= h_u(k_a)h_u = h_u(s)h_u - h_u(q)h_u = S - Q \ , \\
s &= \tfrac{1}{2}(u^k{}_{;m} \cdot g^{ml} - u^l{}_{;m} \cdot g^{mk}) \cdot e_k \wedge e_l \ , \\
q &= \tfrac{1}{2}(T_{mn}{}^k \cdot g^{ml} - T_{mn}{}^l \cdot g^{mk}) \cdot u^n \cdot e_k \wedge e_l \ , \\
S &= h_u(s)h_u \ , \qquad Q = h_u(q)h_u \ .
\end{aligned} \tag{23}
$$

The tensor S is the *torsion-free rotation velocity* tensor, the tensor Q is the *rotation velocity* tensor *induced by the torsion.*

By means of the expressions for σ, ω and θ the deformation velocity tensor d can be decomposed in two parts: d_0 and d_1

$$
d = d_o - d_1 \ , \qquad d_o = {}_sE + S + \frac{1}{n-1}.\theta_o.h_u \ , \qquad d_1 = {}_sP + Q + \frac{1}{n-1}.\theta_1.h_u \ , \tag{24}
$$

where d_o is the *torsion-free deformation velocity* tensor and d_1 is the *deformation velocity* tensor *induced by the torsion.* For the case of V_n-spaces $d_1 = 0$ (${}_sP = 0$, $Q = 0$, $\theta_1 = 0$).

After some calculations, the shear velocity tensor σ and the expansion velocity θ can also be written in the forms

$$
\sigma = \tfrac{1}{2}\{h_u(\nabla_u\overline{g} - \pounds_u\overline{g})h_u - \tfrac{1}{n-1} \cdot (h_u[\nabla_u\overline{g} - \pounds_u\overline{g}]) \cdot h_u\} =
$$

$$
= \tfrac{1}{2}\{h_{i\overline{k}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{\overline{l}j} - \tfrac{1}{n-1} \cdot h_{\overline{kl}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{ij}\} \cdot e^i.e^j \ , \tag{25}
$$

$$
\theta = \frac{1}{2} \cdot h_u[\nabla_u\overline{g} - \pounds_u\overline{g}] = \frac{1}{2} h_{\overline{ij}} \cdot (g^{ij}{}_{;k} \cdot u^k - \pounds_u g^{ij}) \ . \tag{26}
$$

The physical interpretation of the velocity tensors d, σ, ω, and of the invariant θ for the case of V_4-spaces [38], [29], can also be extended for $(\overline{L}_4, g)$-spaces (for more details see [28]). It is easy to be seen that the existence of some kinematic characteristics (${}_sP$, Q, θ_1) depends on the existence of the torsion tensor field. They vanish if it is equal to zero (e.g. in V_n-spaces).

Remark. It should be stressed that the decomposition of the deformation tensor d could not follow from the decomposition of $u^i{}_{;j}$ as this has been done by Ehlers [29] for (pseudo) Riemannian spaces without torsion (V_n-spaces, $n = 4$). In $(\overline{L}_n, g)$-spaces

$$u^i{}_{;j} = \frac{1}{e} \cdot a^i \cdot u_{\overline{j}} + g^{ik} \cdot ({}_sE_{\overline{k}\overline{j}} + S_{\overline{k}\overline{j}} + \frac{1}{n-1} \cdot \theta_0 \cdot h_{\overline{k}\overline{j}}) + + \frac{1}{2 \cdot e} \cdot u^i \cdot (e_{,l} - g_{mn;l} \cdot u^{\overline{m}} \cdot u^{\overline{n}}) \cdot h^{lk} \cdot g_{\overline{k}\overline{j}} \quad . \tag{27}$$

We can now introduce the notion of shear-free and expansion-free flow.

Definition 3. A flow is called a shear-free and expansion-free flow if it has no shear velocity σ and no expansion velocity θ, i.e. a flow is a shear-free and expansion-free flow if $\sigma = 0$ and $\theta = 0$.

The representation of the shear velocity tensor σ and the expansion invariant θ by means of the covariant and Lie derivatives of the contravariant metric tensor $\overline{g}$ give rise to some important conclusions about their vanishing or nonvanishing in a $(\overline{L}_n, g)$-space.

From the structure of σ and θ in the expressions (25) and (26) respectively, it is obviously that if $\pounds_u\overline{g} = \nabla_u\overline{g}$ then $\sigma = 0$ and $\theta = 0$, i.e. the condition $\pounds_u\overline{g} = \nabla_u\overline{g}$ appears as a sufficient conditions for $\sigma = 0$ and $\theta = 0$. On the other side, this condition could be written in the form $\pounds_u g = \nabla_u g$ because of the relations $\pounds_u\overline{g} = -\overline{g}(\pounds_u g)\overline{g}$ and $\nabla_u\overline{g} = -\overline{g}(\nabla_u g)\overline{g}$.

On the basis of the above considerations we could now formulate the following proposition:

Proposition 3. If a metric g in a space with affine connections and metrics [a $(\overline{L}_n, g)$- or a (L_n, g)-space] fulfills the condition

$$\pounds_u\overline{g} = \nabla_u\overline{g} \quad \text{or} \quad \pounds_u g = \nabla_u g \ , \tag{28}$$

then the space admits a non-null shear-free and expansion-free contravariant vector field u.

2.2 Shear-free and expansion-free velocity vector

If we now compare the recurrent relations obtained as a result of the action of the covariant differential operator ∇_u and of the Lie differential operator $\pounds_u$ on the invariant volume element $d\omega$ the question could arise what are the conditions for the equivalence of the action of both the differential operators on $d\omega$, i.e. under which conditions for the vector field u and the metric g we could have the relation

$$\nabla_u(d\omega) = \pounds_u(d\omega) \ , \tag{29}$$

which is equivalent to the relation

$$\overline{g}[\nabla_u g] = \overline{g}[\pounds_u g] \quad \text{or} \quad \overline{g}[\nabla_u g - \pounds_u g] = 0 \quad . \tag{30}$$

What does this relation physically mean? The action of the covariant differential operator ∇_u is determined only on a curve at which u is a tangent vector. The dragging of a contravariant vector ξ along u requires the existence of $\nabla_\xi u$ as the change of the vector u by a transport along ξ [because of the relation $\pounds_u\xi = \nabla_u\xi - \nabla_\xi u - T(u,\xi)$]. The existence of $\nabla_\xi u$ is related to the condition that the components $\xi^i(x^k)$ of the vector $\xi = \xi^i \cdot \partial_i$ should be differentiable functions on neighborhoods of the points of the curve with tangent vector u. This condition is not necessary for a transport of ξ along u. On this ground, the dragging of ξ along

u determines neighborhoods of the points of the curve with tangent vector u. These neighborhoods are moving with the flow along its velocity vector u.

The transport of g is only on the curve and not on the vicinities out of the points of the curve. The action of the Lie differential operator is determined on the vicinities on and out of the points of the curve with tangent vector u on it. The dragging-along of g is on the whole vicinities of the points of the curve and not only along the curve. If the dragging-along u of g is equal to the transport of g along u, then an observer with its worldline as the curve with tangent vector u could not observe any changes in its worldline vicinity different from those who could register on its worldline. The observer will see its surroundings as if they are moving with him.

It is obvious that in the general case, in $(\overline{L}_n, g)$-spaces, a sufficient condition for fulfilling the last relation is the condition

$$\nabla_u g - \pounds_u g = 0 \quad . \tag{31}$$

Let us now discuss in brief the physical interpretation of the last condition.

The change of the length of a vector field u along the curve to which it is a tangent vector field could be written for $(\overline{L}_n, g)$-spaces in the form [33]

$$ul_u = \pm \frac{1}{2 \cdot l_u} \cdot [(\nabla_u g)(u, u) + 2 \cdot g(\nabla_u u, u)] \quad , \qquad l_u \neq 0 \ ,$$

or in the form

$$ul_u = \pm \frac{1}{2 \cdot l_u} \cdot (\pounds_u g)(u, u) \ .$$

Therefore,

$$(\nabla_u g)(u, u) + 2 \cdot g(a, u) = (\pounds_u g)(u, u) \quad , \qquad a = \nabla_u u \quad .$$

If the relation $\nabla_u g = \pounds_u g$ is fulfilled then it follows from the last relation that $g(a, u) = 0$. If the vector field u is interpreted as the velocity of a particle (matter element) then the vector field a is interpreted as its acceleration. Therefore, the condition $\nabla_u g = \pounds_u g$ is a sufficient condition for the acceleration a of a particle to be orthogonal to its velocity u.

Remark. In U_n- and V_n-spaces this condition is automatically fulfilled if u is considered as a normalized vector field [$g(u, u) = e = e_0 = \text{const.} \neq 0$]. In more sophisticated spaces this condition is not fulfilled identically even if u is a normalized vector field because of the relation (see the above expression for ul_u)

$$g(a, u) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u, u)] \ .$$

Remark. The last expression could be interpreted physically as a criteria for the non-orthogonality of the acceleration a and the velocity u in a $(\overline{L}_n, g)$-space. If $e = g(u, u) := \text{const.}$ then $ue = 0$ and the non-metricity ($\nabla_u g \neq 0$) could be considered as a direct criteria for the non-orthogonality of a and u

$$g(a, u) = \frac{1}{2} \cdot (\nabla_u g)(u, u) \ .$$

Therefore, the imposition of the condition $\nabla_u g = \pounds_u g$ restricts the possible motions of a particle with velocity u in a $(\overline{L}_n, g)$-space to motions with acceleration a orthogonal to the velocity u.

On the other side, the Killing equation is usually related to symmetries of the metric tensor g. If a Killing vector exists then the metric, written in appropriate co-ordinates so that the Killing vector is tangent to one of them, will not depend

on this co-ordinate. From physical point of view the Killing equation $\pounds_u g = 0$ for the vector u leads to conservation of the length l_u of u along the same vector, i.e. $\pounds_u g = 0$ is a sufficient condition for preservation of the velocity vector u along the trajectory of a particle moving with velocity u.

The condition (31) is fulfill:

(a) in Riemannian spaces (with or without torsion) [for which $\nabla_u g = 0$ for $\forall u \in T(M)$], when the Killing equation [39]

$$\pounds_u g = 0 \tag{32}$$

is fulfilled for the vector field u.

(b) in Weyl's spaces (with or without torsion) [for which $\nabla_u g = \frac{1}{n} \cdot Q_u \cdot g$ for $\forall u \in T(M)$], when the conformal Killing equation

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \tag{33}$$

is fulfilled for the vector field u.

In $(\overline{L}_n, g)$-spaces, the relation $\nabla_u g - \pounds_u g = 0$ could be written in a co-ordinate basis in the form

$$\begin{aligned} \pounds_u g_{ij} &= g_{ij;k} \cdot u^k + g_{kj} \cdot u^{\overline{k}}{}_{;\underline{i}} + g_{ik} \cdot u^{\overline{k}}{}_{;\underline{j}} + (g_{kj} \cdot T_{l\underline{i}}{}^{\overline{k}} + g_{ik} \cdot T_{l\underline{j}}{}^{\overline{k}}) \cdot u^l = \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (34) \\ &= g_{ij;k} \cdot u^k \ , \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (35) \end{aligned}$$

or in the forms

$$g_{kj} \cdot u^{\overline{k}}{}_{;\underline{i}} + g_{ik} \cdot u^{\overline{k}}{}_{;\underline{j}} + (g_{kj} \cdot T_{l\underline{i}}{}^{\overline{k}} + g_{ik} \cdot T_{l\underline{j}}{}^{\overline{k}}) \cdot u^l = 0 \quad , \tag{36}$$

$$g_{kj} \cdot (u^{\overline{k}}{}_{;\underline{i}} - T_{\underline{i}l}{}^{\overline{k}} \cdot u^l) + g_{ik} \cdot (u^{\overline{k}}{}_{;\underline{j}} - T_{\underline{j}l}{}^{\overline{k}} \cdot u^l) = 0 \ , \tag{37}$$

where

$$T_{\underline{i}l}{}^{\overline{k}} := f_i{}^m \cdot T_{ml}{}^n \cdot f^k{}_n, \qquad\qquad u^{\overline{k}}{}_{;\underline{j}} := f^k{}_l \cdot u^l{}_{;m} \cdot f_j{}^m.$$

The equation (37) could be called "generalized conformal Killing equation" for the vector field u in the case $\pounds_u g = \nabla_u g$.

After multiplication of the last expression, equivalent to $\pounds_u g_{ij} = g_{ij;k} \cdot u^k$, with $u^{\overline{i}}$ and $g^{m\overline{j}}$ and summation over $\overline{i}$ and $\overline{j}$ (and then changing the index m with i) we can find the equation for the vector field u in a co-ordinate basis

$$u^i{}_{;j} \cdot u^j + g_{\overline{lk}} \cdot u^l \cdot (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} = 0 \quad , \tag{38}$$

or as an equation for the acceleration $a = a^i \cdot \partial_i$ with

$$a^i = -g_{\overline{lk}} \cdot u^l \cdot (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} = -g_{\overline{lk}} \cdot u^l \cdot k^{ki} \quad , \tag{39}$$

where

$$k^{ki} = (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} \ . \tag{40}$$

On the other side, from the equation (37) it is obvious that a sufficient condition for fulfilling the equation (37) is the condition for u^i

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \qquad \text{or} \qquad u^k{}_{;j} = T_{jl}{}^k \cdot u^l \quad . \tag{41}$$

From the last expression, it follows that if the vector u fulfills this condition it should be an auto-parallel vector field since

$$u^i{}_{;j} \cdot u^j = a^i = 0 \quad . \tag{42}$$

If (41) is fulfilled, then the following relations are valid:

$$u^m \cdot R^i{}_{mkl} = -(T_{ml}{}^i{}_{;k} - T_{mk}{}^i{}_{;l} + T_{ml}{}^n \cdot T_{nk}{}^i -$$

$$-T_{mk}{}^n \cdot T_{nl}{}^i + T_{kl}{}^n \cdot T_{mn}{}^i) \cdot u^m \quad ,$$

$$R_{mk} \cdot u^m \cdot u^k = g^l_i \cdot R^i{}_{mkl} \cdot u^m \cdot u^k = T_{lm}{}^l{}_{;k} \cdot u^k \cdot u^m \quad .$$

On the basis of the above relations, we can prove the preposition:

Proposition 4. If a contravariant non-null vector field u in a space with affine connections and metrics [a $(\overline{L}_n, g)$- or a (L_n, g)-space] fulfills the equation (41)

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \qquad \text{or} \qquad u^k{}_{;j} = T_{jl}{}^k \cdot u^l \ ,$$

then this vector field is an auto-parallel shear-free and expansion-free vector field.

Let us now consider a Weyl's space admitting the condition (31).

For Weyl's spaces ($\nabla_u g = \frac{1}{n} \cdot Q_u \cdot g$) the condition (31) degenerate in the condition for the existence of a conformal Killing vector u

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \ . \tag{43}$$

Therefore, we can prove the following propositions:

Proposition 5. If a contravariant non-null vector field in a Weyl's space fulfills a conformal Killing equation of the type

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \ , \tag{44}$$

then this conformal vector field u is also a shear-free and expansion-free vector field.

Proposition 6. If a contravariant non-null vector field u in a Weyl's space fulfills the equation (41)

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \quad ,$$

then it is an auto-parallel, shear-free and expansion-free conformal Killing vector field.

The auto-parallel equation (42) for the vector field u is interpreted as an equation of motion for a free spinless test particle in spaces with affine connection and metrics [22], [23]. Let us now consider this equation more closely in Weyl's spaces.

3 Auto-parallel equation in Weyl's spaces as an equation for a free moving spinless test particles

Every (covariant) gravitational theory in spaces with affine connections and metrics should obey a condition (equation) for description of the motion of a free moving spinless test particle.

The notion of a (spinless) test particle is usually related to a material point moving in an external field or in space-time without changing the characteristics of the external field or of the space-time. This means that the dynamical characteristics of the test particle does not generate additional influence on the particle's motion under external conditions. If a (spinless) test particle is moving in the space-time it does not act on its geometric properties related in some gravitational theories to the properties of other material points, distributions and dynamical systems on the basis of field equations. These field equations describe the evolution of the physical structures in the space-time and vice versa: the evolution of the space-time under the existence of physical systems in it. Under these conditions, it is assumed that

the motion of a (spinless) test particle depends explicitly only on the geometric characteristics of the mathematical model describing the space-time and only implicitly on the field equations determining the geometric characteristics of the space-time. On this basis, we can consider all characteristics of a space-time model as given and ignore the structure of a concrete field theory (field equations) determining them.

In (pseudo) Riemannian spaces without torsion (V_n-spaces) ($n = 4$) the motion of a free spinless particle is described by the use of a geodesic line identical in this type of spaces with an auto-parallel trajectory. It is *a belief* of some authors that a geodesic line should also be the trajectory of a free moving spinless test particle in (L_n, g)- and $(\overline{L}_n, g)$-spaces where geodesics are different from auto-parallel trajectories. The reasons for this belief are the different constructions of theories where the difference between geodesics and auto-parallels has been seen as related to the existence of forces generated by the space-time. Recently, it has been proved [23] that a free spinless test particle could move on an auto-parallel trajectory in a (L_n, g)- or in a $(\overline{L}_n, g)$-spaces. This fact is based on the principle of equivalence [9]÷[16] and could not be ignored in physical investigations [40]. Moreover, the presence of an additional force term (defined by non-metricity) does not mean that a particle has an additional charge (see, for instance, inertial forces in V_n- and E_n-spaces) because this term could be removed by the use of an appropriate extended differential operator generating a new affine connection and a new frame of reference in space-time [40].

Usually the following definition of a free moving test particle in a space with affine connections and metrics [and especially in (pseudo) Riemannian spaces without torsion] is introduced [41]:

Definition 4. A free spinless test particle is a material point with rest mass (density) ρ, velocity u (as tangent vector u to its trajectory) and momentum (density) $p := \rho \cdot u$ with the following properties:

(a) The momentum density p does not change its direction along the world line of the material point, i.e. the vector p fulfills the recurrent condition $\nabla_u p = f \cdot p$, or the condition $\nabla_u p = 0$, as conditions for parallel transport along u.

(b) The momentum density p does not change its length $l_p =\mid g(p,p) \mid^{1/2}$ along the world line of the material point, i.e. p fulfills the condition $ul_p = 0$.

The change of the length of a vector field p along a vector field u in a $(\overline{L}_n, g)$-space could be found in the form [33]

$$ul_p = \pm \frac{1}{2 \cdot l_p} \cdot [(\nabla_u g)(p,p) + 2 \cdot g(\nabla_u p, p)] \quad , \qquad l_p \neq 0 \quad . \tag{45}$$

Remark. The basis for this definition is the consideration of the notion of momentum density in a classical field theory in spaces with affine connections and metrics [26], [27]. A material point (mass element) is characterized by its rest mass density ρ and velocity u. A material point (particle) which does not interact with its surroundings should have energy-momentum tensor G of the type $(G)\overline{g} = u \otimes p$ with $p = \rho \cdot u$ and p should not change along its trajectory with tangent vector u.

Let us now consider the two conditions (a) and (b) for p separately to each other.

(a) If we write p in its explicit form $p = \rho \cdot u$, then the condition for a parallel transport of p along u could be written as

$$\nabla_u u = [f - u(\log \rho)] \cdot u \ , \tag{46}$$

with

$$f = u(\log \rho) + \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u,u)] \quad , \qquad e = g(u,u) \neq 0 \ , \tag{47}$$

and

$$\nabla_u u = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u,u)] \cdot u \ . \tag{(a)}$$

In the special case of (pseudo) Riemannian spaces (V_n- or U_n- spaces), where $\nabla_u g = 0$, $e = \text{const.} \neq 0$, $\rho = \text{const.}$, $f = 0$, it follows that $\nabla_u p = 0$, and $\nabla_u u = 0$. At the same time, $ul_p = 0$. The parallel equation $\nabla_u p = 0$ has as a corollary the preservation of the length l_p of p along u. This is not the case if a space is not a (pseudo) Riemannian space.

(b) The conservation of the momentum density p along the trajectory of the particles $[ul_p = 0]$ requires the transport of p on this trajectory to be of the type of a Fermi-Walker transport, i.e. p should obey an equation of the type [17], [19]

$$\nabla_u p = \overline{g}({}^F\omega - \frac{1}{2} \cdot \nabla_u g)(p) = \overline{g}[{}^F\omega(p)] - \frac{1}{2} \cdot \overline{g}(\nabla_u g)(p) \ , \tag{48}$$

where ${}^F\omega \in \Lambda^2(M)$ is an antisymmetric tensor of 2nd rank. For a free particle it could be related to the rotation velocity tensor (23) of the velocity u, i.e. ${}^F\omega := \omega$. Then $\omega(p) = 0$ [because of $\omega(\rho \cdot u) = \rho \cdot (\omega(u)) = 0$] and we have for $\nabla_u p$

$$\nabla_u p = -\frac{1}{2} \cdot \overline{g}(\nabla_u g)(p) \ , \qquad ul_p = 0 \ . \tag{49}$$

For the vector field u follows the corresponding condition

$$\nabla_u u = -\{[u(\log \rho)] \cdot u + \frac{1}{2} \cdot \overline{g}(\nabla_u g)(u)\} \ . \tag{(b)}$$

Therefore, for u we have two equations as corollaries from the requirements for the momentum density p: equation (a) which follows from the condition for preservation of the direction of the momentum density p, and equation (b) which follows from the condition for preservation of the length l_p of the momentum density p.

(a) The first equation (a) for u is the auto-parallel equation in its non-canonical form. It does not depend on the rest mass density ρ of the particle. From the equation, it follows that the necessary and sufficient condition for a spinless test particle to move in a space with affine connections and metrics on a trajectory described by the auto-parallel equation in its canonical form ($\nabla_u u = 0$) is the condition

$$[ue - (\nabla_u g)(u,u)] \cdot u = 0 \quad , \qquad e \neq 0 \ . \tag{50}$$

Since $g(u,u) = e \neq 0$, after contracting the equation with $g(u)$, we obtain the condition

$$ue - (\nabla_u g)(u,u) = 0 \ , \qquad \text{or} \qquad ue = (\nabla_u g)(u,u) \quad . \tag{51}$$

This condition determines how the length of the vector u should change with respect to the change of the metric g along u if u should be an auto-parallel vector field with $\nabla_u u = 0$.

If we consider a Weyl's space as a model of space-time, this condition will take the form

$$ue = \frac{1}{n} \cdot Q_u \cdot e \quad \text{, or} \qquad u(\log e) = \frac{1}{n} \cdot Q_u \quad , \tag{52}$$

leading to the relation for e

$$e = e_0 \cdot \exp(\frac{1}{n} \cdot \int Q_u \cdot d\tau) \quad , \qquad e_0 = \text{const.}, \tag{53}$$

where $u = d/d\tau$ and $\tau = \tau(x^k)$ is the canonical parameter of the trajectory of the particle.

If Q_u is constructed by the use of a dilaton field $\overline{\varphi}$ as $Q_u = -d\overline{\varphi}/d\tau$, then e would change under the condition

$$e = e_0 \cdot \exp(-\frac{1}{n} \cdot \overline{\varphi}) \ . \tag{54}$$

The dilaton field $\overline{\varphi}$ could be represented by means of e in the form

$$\overline{\varphi} = -n \cdot \log(\frac{e}{e_0}) \ . \tag{55}$$

Therefore, the dilaton field $\overline{\varphi}$ takes the role of a length scaling factor for the velocity of a test particle.

(b) From the second equation (b) for u, it follows that a necessary and sufficient condition for a free spinless test particle to move in a space with affine connections and metrics on a trajectory, described by the auto-parallel equation in its canonical form ($\nabla_u u = 0$), is the condition

$$[u(\log \rho)] \cdot u = -\frac{1}{2} \cdot \overline{g}(\nabla_u g)(u) \quad . \tag{56}$$

Since $g(u, u) = e \neq 0$, after contracting the last equation with $g(u)$ we obtain the condition

$$u(\log \rho) \cdot e = -\frac{1}{2} \cdot \overline{g}(\nabla_u g)(u)[g(u)] = -\frac{1}{2} \cdot (\nabla_u g)(u, u) \quad ,$$

or

$$u(\log \rho) = -\frac{1}{2 \cdot e} \cdot (\nabla_u g)(u, u) \ . \tag{57}$$

For $u = d/d\tau$, it follows the equation

$$\frac{d}{d\tau}(\log \rho) = -\frac{1}{2 \cdot e} \cdot (\nabla_u g)(u, u) \ ,$$

with the solution for $\rho(x^k(\tau))$

$$\rho = \rho_0 \cdot \exp(-\frac{1}{2} \cdot \int \frac{1}{e} \cdot (\nabla_u g)(u, u) \cdot d\tau) \quad .$$

The last condition is for the rest mass density ρ which it has to obey if the particle should move on an auto-parallel trajectory or if we observe the motion of a particle as a free motion in the corresponding space considered as a mathematical model of the space-time.

If we consider a Weyl's space as a model of the space-time, the condition (56) will take the form

$$[u(\log \rho)] \cdot u = -\frac{1}{2 \cdot n} \cdot Q_u \cdot u \ , \tag{58}$$

or the form

$$[u(\log \rho) + \frac{1}{2 \cdot n} \cdot Q_u] \cdot u = 0 \ .$$

Since $g(u, u) = e \neq 0$, after contracting the last equation with $g(u)$ we obtain the condition

$$u(\log \rho) + \frac{1}{2 \cdot n} \cdot Q_u = 0 \quad . \tag{59}$$

Therefore, the rest mass density ρ should change on the auto-parallel trajectory of the particle as

$$\rho = \rho_0 \cdot \exp\left[-\frac{1}{2 \cdot n} \cdot \int Q_u \cdot d\tau\right] \ , \qquad \rho_0 = \text{const.} \tag{60}$$

Furthermore, if Q_u is constructed by the use of a dilaton field $\overline{\varphi}$ as $Q_u = -d\overline{\varphi}/d\tau$, then ρ would change under the condition

$$\rho = \rho_0 \cdot \exp\left[\frac{1}{2 \cdot n} \cdot \int \frac{d\overline{\varphi}}{d\tau} \cdot d\tau\right] = \rho = \rho_0 \cdot \exp(\frac{1}{2 \cdot n} \cdot \overline{\varphi}) \ . \tag{61}$$

The dilaton field $\overline{\varphi}$ could be represented by means of ρ in the form

$$\overline{\varphi} = 2 \cdot n \cdot (\log \frac{\rho}{\rho_0}) \ . \tag{62}$$

Therefore, the dilaton field $\overline{\varphi}$ takes the role of a mass density scaling factor for the rest mass density of a test particle. This is another physical interpretation as the interpretation used by other authors as mass field, pure geometric gauge field and etc.

Since $\overline{\varphi} = -n \cdot \log(e/e_0) = 2n \cdot \log(\rho/\rho_0)$, a relation between e and ρ follows in the form

$$\rho^2 \cdot e = \rho_0^2 \cdot e_0 = \text{ const. } = l_p^2 \ , \tag{63}$$

which is exactly the condition (b) of the definition for a free moving spinless test particle.

4 Conclusion

In the present paper the conditions are found under which a space with affine connections and metrics and especially a Weyl's space admit shear-free and expansion-free non-null vector fields as velocities of flows or of test particles. In a Weyl's space the vector fields appear as conformal Killing vector fields. In such type of spaces only the rotation velocity is not vanishing. This fact could be used as a theoretical basis for models in continuous media mechanics and in the modern gravitational theories, where a rotation velocity could play an important role. Further, necessary and sufficient conditions are found under which a free spinless test particle could move in spaces with affine connections and metrics on an auto-parallel curve. In Weyl's spaces with Weyl's covector, constructed by the use of a dilaton field, the dilaton field appears as a scaling factor for the rest mass density as well as for the velocity of the test particle. The last fact leads to a new physical interpretation of a dilaton field in classical field theories over spaces with affine connections and metrics and especially over Weyl's spaces. The obtained results could be used in constructing local criteria for experimental check-up of the existence of non-metricity and torsion in realistic models of space-time.

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